# Hierarchical Polynomial Approximation

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# Outline

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# Two-Level Polynomial Approximations

The problem: **After scaling, values of $f(0)$, $f(1)$, $f(2)$, ..., $f(T_I - 1)$ on a large interval $I$, of width $T_I$?**

- On $I$, $f$ is approximated by a degree-$d$ polynomial $P$, with an approximation error bounded by $\epsilon$.
- $I$ is split into sub-intervals $J_0$, $J_1$, $J_2$, ... of width $T_J$, where $J_k = J_{k-1} + T_J$.
- On each of the $J_k$, we are looking for an approximation $Q_k$ of degree $\delta < d$.

$$\begin{aligned} P_k(X) &= P(X + k \cdot T_J) = a_0(k) + a_1(k) \cdot X + a_2(k) \cdot \frac{X(X-1)}{2} + \\ & \cdots + a_d(k) \cdot \frac{X(X-1)(X-2)\cdots(X-d+1)}{d!} \end{aligned}$$

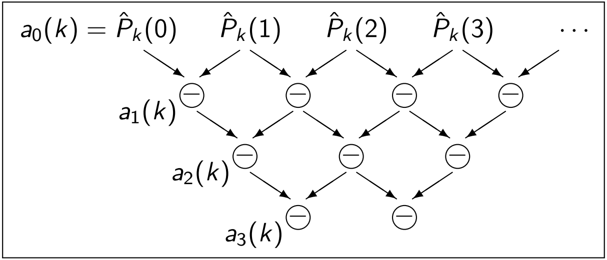
$$\begin{aligned} Q_k(X) &= a_0(k) + a_1(k) \cdot X + a_2(k) \cdot \frac{X(X-1)}{2} + \cdots + \\ & a_\delta(k) \cdot \frac{X(X-1)(X-2)\cdots(X-\delta+1)}{\delta!} = \sum_{i=0}^{\delta} a_i(k) \cdot \binom{X}{i} \end{aligned}$$

# Computation of the Coefficients of $a_i(k)$

The problem: **Compute the values of $a_i(k)$.**

- For each $i \in [\![0, \delta]\!]$, $a_i(k)$ is a degree-$(d - i)$ polynomial function of $k$.
- We can compute the $a_i(k)$ from the $d - i$ consecutive values $P_k(0)$, $P_k(1)$, $P_k(2)$, …, $P_k(d - i - 1)$ for the first values of $k$; the following $a_i(k)$'s are computed with the *difference table method*.
- Let $P_k(u)$ be the "true" value, and $\hat{P}_k(u)$ be the "computed" value.

For $0 \le k \le d - i$:



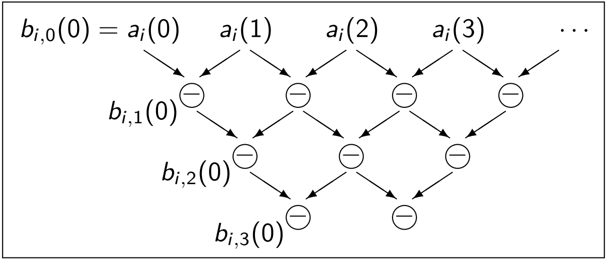

Let $e$ be an upper bound on $\left|P_k(u) - \hat{P}_k(u)\right|$.

With exact subtractions, $a_i(k)$ will be known with an error $\le 2^i e$.

# Computation of the Coefficients of $a_i(k)$ [2]

Define values $b_{i,j}(k)$ as

$$
\begin{aligned}
a_i(k) &= b_{i,0}(k) + b_{i,1}(k)\cdot k + b_{i,2}(k)\cdot \frac{k(k-1)}{2} + \cdots + \\
&\quad b_{i,d-i}(k)\cdot \frac{k(k-1)(k-2)\cdots(k-d+i+1)}{(d-i)!} = \sum_{j=0}^{d-i} b_{i,j}(k)\cdot \binom{k}{j}
\end{aligned}
$$




The term $b_{i,j}(0)$ is known with an error $\leq 2^{i+j}e$.

The $a_i(k)$ will be computed with additions from these coefficients $b_{i,j}(0)$...

# Computation of Consecutive Values of a Polynomial

Let $Q(X+k) = \sum_{i=0}^{\delta} q_i(k) \cdot \binom{X}{i}$ of degree (at most) $\delta$, where the $q_i(k)$'s are assumed to be the exact coefficients.

The problem: **Knowing the initial coefficients $q_i(0)$ of $Q(X)$, compute the consecutive values $Q(0)$, $Q(1)$, $Q(2)$, ...**

We have:
$$\begin{cases} q_0(k) = q_0(k-1) + q_1(k-1) \\ q_1(k) = q_1(k-1) + q_2(k-1) \\ \quad\vdots \\ q_{\delta-1}(k) = q_{\delta-1}(k-1) + q_\delta(k-1) \end{cases}$$

$q_\delta$ being a constant, and $Q(k) = q_0(k)$.

The coefficients $q_i(k)$ will be represented by $\hat{q}_i(k)$ with $n_i$ bits after the fractional point, i.e. on $n_i$ bits as we are interested in the values modulo 1, and an initial error of one ulp: $u_i = 2^{-n_i}$. Since $q_i(k)$ depends on $q_{i+1}(k-1)$, we assume that $(n_i)$ is increasing: $n_i \leq n_{i+1}$. And $n_\delta = n_{\delta-1}$ for the constant coefficient $q_\delta$ (using more precision would be useless).

# Computation of Consecutive Values of a Polynomial [2]

Formally: $n_i \le n_{i+1}$, $u_i = 2^{-n_i}$, $\hat{q}_i(k) \in u_i\mathbb{Z}$ or $u_i\mathbb{Z}/\mathbb{Z}$, and $|\hat{q}_i(0) - q_i(0)| \le u_i$.

Basic iteration: $$\begin{cases} \hat{q}_0(k) &= \hat{q}_0(k-1) &+ \circ(\hat{q}_1(k-1)) \\ \hat{q}_1(k) &= \hat{q}_1(k-1) &+ \circ(\hat{q}_2(k-1)) \\ & \vdots & \\ \hat{q}_{\delta-2}(k) &= \hat{q}_{\delta-2}(k-1) &+ \circ(\hat{q}_{\delta-1}(k-1)) \\ \hat{q}_{\delta-1}(k) &= \hat{q}_{\delta-1}(k-1) &+ \hat{q}_\delta \end{cases}$$

where $|\hat{q}_\delta - q_\delta| \le u_{\delta-1}$ and $\circ$ denotes the truncation of the value to the precision of the result (said otherwise, when doing an addition, the trailing bits of the more precise value are ignored).

Let $\epsilon_i(k)$ be a bound on the error on $q_i(k)$, i.e. $|\hat{q}_i(k) - q_i(k)| \le \epsilon_i(k)$.
Initially, $\epsilon_i(0) = u_i$ for $0 \le i \le \delta - 1$.

We have: $\epsilon_i(k) = \epsilon_i(k-1) + \epsilon_{i+1}(k-1) + u_i$ for $0 \le i \le \delta - 1$,
with $\epsilon_\delta(0) = 0$ in order to satisfy $\epsilon_{\delta-1}(k) = \epsilon_{\delta-1}(k-1) + u_{\delta-1}$.

We can prove by induction that $\epsilon_i(k) = \sum_{j=i}^{\delta-1} u_j \cdot \binom{k+1}{j-i+1}$.

# Computation of Consecutive Values of a Polynomial [3]

Proof by induction that $\epsilon_i(k) = \sum_{j=i}^{\delta-1} u_j \cdot \binom{k+1}{j-i+1}$.

This is true for $k = 0$ and for $i = \delta$, and

$$\begin{aligned}
\epsilon_i(k-1) + \epsilon_{i+1}(k-1) + u_i &= u_i + \sum_{j=i+1}^{\delta-1} u_j \cdot \binom{k}{j-i} + \sum_{j=i}^{\delta-1} u_j \cdot \binom{k}{j-i+1} \\
&= \sum_{j=i}^{\delta-1} u_j \cdot \left[\binom{k}{j-i} + \binom{k}{j-i+1}\right] \\
&= \sum_{j=i}^{\delta-1} u_j \cdot \binom{k+1}{j-i+1} = \epsilon_i(k)
\end{aligned}$$

# Computation of Consecutive Values of a Polynomial [4]

If $\ell$ denotes the length of the interval (the number of values), i.e. $0 \le k < \ell$, then the error is bounded by:

$$\max_{0 \le k < \ell} \epsilon_0(k) \le \sum_{i=0}^{\delta-1} u_i \cdot \binom{\ell}{i+1} = \sum_{i=0}^{\delta-1} 2^{-n_i} \cdot \binom{\ell}{i+1}.$$

The values of $n_i$ will be determined so that this error bound is less than some given error bound $E$. This can be done by determining $n_0$, then $n_1$, then $n_2$, and so on, each time by taking the smallest $n_i$ (multiple of the word size) such that $2^{-n_i} \cdot \binom{\ell}{i+1}$ is less than a fraction of the remaining error bound.

Note: if $n_{i+1} = n_i$, one could have taken $\epsilon'_i(k) = \epsilon_{i+1}(k)$, but the gain would probably be low (since $\epsilon_{i+1}(k) \sim k \cdot u_{i+1} = k \cdot u_i$ at least) and the formulas would probably be much more complicated.

# Error Bound on the Approximation of $P_k$ by $Q_k$

On the interval $J_k$: $P_k(m) - Q_k(m) = \sum_{i=\delta+1}^{d} a_i(k) \cdot \binom{m}{i}$

with $a_i(k) = \Delta^i P(kT_J) = \sum_{j=i}^{d} a_j(0) \cdot \binom{kT_J}{j-i}$.

Thus

$$P_k(m) - Q_k(m) = \sum_{j=\delta+1}^{d} a_j(0) \sum_{i=\delta+1}^{j} \binom{kT_J}{j-i} \binom{m}{i}$$

with $0 \le m \le T_J - 1$ and $0 \le kT_J \le T_I - T_J$. This gives the following error bound on the approximation of $P_k$ by $Q_k$:

$$\sum_{j=\delta+1}^{d} |a_j(0)| \sum_{i=\delta+1}^{j} \binom{T_I - T_J}{j-i} \binom{T_J - 1}{i}$$

but one can get better dynamical bounds by considering the sign of $a_j(0)$.

# Summary for the Search for HR-Cases

1. Determine the exponent range of $f$; $f$ will be scaled by a power of the radix, so that we are interested in the values modulo 1: $\tilde{f} = \beta^s \cdot f$.
2. Determine the admissible final error bound $\varepsilon_0$ on $\tilde{f}$.
3. Assume we have a degree-$d$ polynomial approximation $P(m)$ to $\tilde{f}(x)$ with an error bound $\varepsilon_f$, where $x = x_0 + m \cdot d_x$ (the evaluation error of the polynomial is not taken into account here).
4. $P_k(X) = P(X + kT_J)$ will be approximated by degree-$\delta$ polynomials $Q_k$. The coefficients $a_i(k)$ of $Q_k$ are in fact polynomials in $k$ of degree $d - i$, where $0 \leq i \leq \delta$. The initial coefficients $b_{i,j}(0)$ of these polynomials $a_i(k)$ in the binomial base are computed with an error bounded by $2^{i+j}e$, where $e$ is a bound on the evaluation error of the polynomial. For the ulp error condition, we want $2^{i+j}e \leq 2^{-n_j-1}$, thus $e \leq 2^{-i-j-n_j-1}$, where the values of the $n_j$'s can be determined so that the error on $a_i(k)$ is less than some given bound:

$$\sum_{j=0}^{d-i-1} 2^{-n_j} \cdot \binom{T_I/T_J}{j+1} \leq E_i$$